

### Abstract

The notions length of a vector field and cosine of the angle between two vector fields are determined by means of the metrics and the corresponding vector fields over a differentiable manifold with contravariant and covariant affine connections and metrics  $[(\bar{L}_n, g)$ -spaces]. The change of the length of a vector field and the cosine of the angle between two vector fields along a contravariant vector field are found.

# $(\overline{L}_n, g)$ -spaces. Length of a vector and angle between two vectors

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## 1 Introduction

In previous papers [1], [2] the notions of contravariant and covariant affine connections are considered for contravariant and covariant tensor fields over differentiable manifolds. It was shown that these two different (not only by sign) connections can be introduced by means of changing the canonical definition of the bases of dual vector spaces (respectively of dual vector fields). The deviation operator and its applications to deviation equations over differentiable manifolds with contravariant and covariant affine connections [the s.c.  $(\overline{L}_n, g)$ -spaces] are investigated.

In the Einstein theory of gravitation (ETG) kinematic notions related to the notion relative velocity such as shear velocity tensor (shear velocity, shear)  $\sigma$ , rotation velocity tensor (rotation velocity, rotation)  $\omega$  and expansion velocity (expansion)  $\theta$ , are used in finding solutions of special types of Einstein's field equations and in the description of the properties of the (pseudo) Riemannian spaces without torsion ( $V_n$ -spaces). By means of these notions a classification of  $V_n$ -spaces, admitting special types of geodesic vector fields, has been proposed [3]. The same kinematic characteristics are also necessary for description of the projections of the Riemannian (curvature) tensor and the Ricci tensor along a non-isotropic (non-null) vector field [4], and in obtaining and using the Raychaudhuri identity [5], [6] in  $V_n$ -spaces.

The kinematic characteristics connected with the notion relative velocity can be generalized for vector fields over differentiable manifolds with contravariant and covariant affine connections and metrics  $[(\overline{L}_n, g)$ -spaces] so that in the case of  $(L_n, g)$ - and  $V_n$ -spaces [as a special case of  $(\overline{L}_n, g)$ -spaces], and for normalized non-isotropic vector fields these characteristics are the same as those introduced in the ETG. In an analogous way as in the case of the kinematic characteristics, related to the notion of relative velocity, it is possible to introduce kinematic characteristics, related to the notion of relative acceleration such as shear acceleration tensor (shear acceleration), rotation acceleration tensor (rotation acceleration) and expansion acceleration [7].

The corresponding for  $(\overline{L}_n, g)$ -spaces notions of relative velocity and relative acceleration are considered in [2]. By means of these kinematic characteristics,

several other types of notions such as shear velocity, shear acceleration, rotation velocity, rotation acceleration, expansion velocity and expansion acceleration are investigated. The connections between the kinematic characteristics related to the relative acceleration and these related to the relative velocity are also found. The auto-parallel vector fields in  $(\bar{L}_n, g)$ -spaces are classified on the basis of the kinematic characteristics. The generalizations compared with those in  $(L_n, g)$ -spaces (differentiable manifold with affine connection and metric) appear only in the explicit forms of the expressions, written in a corresponding basis (or in other words - only in index forms).

In this paper the basic notions of the length of a vector field and the cosine of the angle between two vector fields are considered over  $(\bar{L}_n, g)$ -spaces as well as their changes along an other vector field. These notions are necessary for introduction and consideration of different types of transports in  $(\bar{L}_n, g)$ -spaces such as Fermi-Walker transports [8] and conformal transports [9].

## 2 Length of a contravariant vector field

The *square of the length of a contravariant vector field*  $u$  is determined by means of the covariant metric tensor  $g$  as

$$u^2 = \pm |u|^2 = g(u, u), \quad u \in T(M), \quad |u| \geq 0. \quad (1)$$

**Definition 1** *The length of a contravariant vector field  $u$  is the positive square root of the absolute value of the square of the length of this field, i.e.*

$$l_u = |u| = |g(u, u)|^{\frac{1}{2}}, \quad l_u(x) = |u_x| = |u(x)|, \quad x \in M, \quad (2)$$

where  $l_u$  is the length of the contravariant vector field  $u$  and  $l_u(x)$  is the length of the contravariant vector  $u(x)$  in a point  $x \in M$ .

With respect to their lengths, the contravariant vector fields can be divided in two classes: *null-* or *isotropic* vector fields ( $l_u = 0$ ) and *non-null* or *non-isotropic* vector fields ( $l_u \neq 0$ ). In the cases of a *positive definite covariant metric*  $g$  ( $\text{Sgn } g = n, \dim M = n$ ) the isotropic vector field is identically equal to zero ( $u \equiv 0, u^\alpha \equiv 0$ ). In the cases of an *indefinite covariant metric*  $g$  ( $\text{Sgn } g < n$  or  $\text{Sgn } g > -n, \dim M = n$ ) the isotropic vector field with equal to zero length can have different from zero components in an arbitrary basis, i.e. it is not identically equal to zero in the points, where it has been defined.

The changes of the length of a contravariant vector field under the action of the covariant differential operator is determined on one side, by the action of the covariant operator on a function (here the length  $l_u$ ) over the manifold and on the other - by the structure of the length  $l_u$  itself and by the commutation relation between the covariant operator and the contraction operator

$$\begin{aligned} \nabla_\xi u^2 &= \pm \nabla_\xi (l_u^2) = \nabla_\xi [g(u, u)] = \xi[g(u, u)] = \pm \xi(l_u^2) = \\ &= \pm 2l_u(\xi l_u) = (\nabla_\xi g)(u, u) + 2g(\nabla_\xi u, u), \end{aligned} \quad (3)$$

from where it follows

$$\xi l_u = \pm \frac{1}{2l_u} [(\nabla_\xi g)(u, u) + 2g(\nabla_\xi u, u)] , \quad l_u \neq 0 . \quad (4)$$

In the case of an null (isotropic) contravariant vector field  $u$ ,  $g(u, u) = 0$ , there is a relation between the covariant derivative  $\nabla_\xi g$  of the covariant metric tensor  $g$  and the covariant derivative  $\nabla_\xi u$  of the vector field  $u$

$$(\nabla_\xi g)(u, u) = -2g(\nabla_\xi u, u) . \quad (5)$$

If the contravariant vector field  $u$  is transported parallel along the contravariant vector field  $\xi$ , the change of the length of  $u$  obeys the condition

$$\xi l_u = \pm \frac{1}{2l_u} (\nabla_\xi g)(u, u) , \quad \nabla_\xi u = 0 , \quad l_u \neq 0 , \quad (6)$$

and if  $l_u = 0$  and  $\nabla_\xi u = 0$ , then the condition  $(\nabla_\xi g)(u, u) = 0$  follows for the covariant metric  $g$  or for the vector field  $\xi$ .

One of the essential characteristics of the different types of transport is their influence on the change of the length of a contravariant vector field due to the action of the covariant differential operator. In the case of manifolds with affine connections and metric which allow different type of transports of the covariant metric, there are transports under which the length of a contravariant vector field does not change.

The change of the length of a contravariant vector field under the action of the Lie differential operator, i. e. the change of length under draggings-along, can be described in an analogous way as in the cases of transports

$$\begin{aligned} \mathcal{L}_\xi u^2 &= \pm \mathcal{L}_\xi |u|^2 = \mathcal{L}_\xi [g(u, u)] = \pm 2l_u (\xi l_u) = \\ &= (\mathcal{L}_\xi g)(u, u) + 2g(\mathcal{L}_\xi u, u) , \end{aligned} \quad (7)$$

from where it follows

$$\begin{aligned} l_u \neq 0 : \xi l_u &= \pm \frac{1}{2l_u} [(\mathcal{L}_\xi g)(u, u) + 2g(\mathcal{L}_\xi u, u)] , \\ l_u = 0 : (\mathcal{L}_\xi g)(u, u) &= -2g(\mathcal{L}_\xi u, u) . \end{aligned} \quad (8)$$

For parallel dragging of  $u$  along  $\xi$  ( $\mathcal{L}_\xi u = 0$ ), the change of  $l_u$  can be written in the form

$$\begin{aligned} l_u \neq 0 : \xi l_u &= \pm \frac{1}{2l_u} (\mathcal{L}_\xi g)(u, u) , \\ l_u = 0 : (\mathcal{L}_\xi g)(u, u) &= 0 . \end{aligned} \quad (9)$$

One of the main characteristics of the different draggings-along of a covariant metric  $g$  is the change of length of a contravariant vector field under draggings-along. The different types of draggings-along of  $g$  induce different changes of the length of a given contravariant vector field.

The length change of contravariant vector fields under different types of "transport" and "draggings-along" of the covariant metric  $g$  has been used in mathematical models of physical systems, described by means of differentiable manifolds with affine connections and metric. The change of length has also been used for giving certain properties and characteristics with physical interpretation of these systems.

### 3 Cosine of the angle between two contravariant vector fields

The cosine of the angle between two contravariant vector fields is determined by means of the scalar product of both vector fields and their lengths. From the relation

$$g(u, v) = |u| \cdot |v| \cdot \cos(u, v) = l_u \cdot l_v \cdot \cos(u, v) , \quad (10)$$

the definition for the cosine between the vector fields  $u$  and  $v$  can be written as

$$\cos(u, v) = \frac{1}{l_u \cdot l_v} \cdot g(u, v) , \quad u, v \in T(M) , \quad l_u \neq 0 , \quad l_v \neq 0 . \quad (11)$$

From the definition it follows, that two non-isotropic contravariant vector fields  $u$  and  $v$  are *orthogonal* to each other, if their scalar product is equal to zero, i. e.

$$u \perp v : g(u, v) = 0 : \cos(u, v) = 0 , \quad u, v \in T(M) . \quad (12)$$

The notion cosine between two isotropic contravariant vector fields or between one non-isotropic and one isotropic contravariant vector field cannot be introduced by means of the above definition.

**Remark 1** *If one assumes that (10) is also fulfilled for the scalar product of two isotropic (null) vector fields (or for one non-isotropic and one isotropic vector field), then from (10) and (12) it could be fixed that: (a) two isotropic contravariant vector fields are always orthogonal to each other; (b) every isotropic contravariant vector field is orthogonal to each non-isotropic contravariant vector field. The last statement leads to the condition  $g(u, v) = 0$ , ( $l_u = 0$ ,  $l_v \neq 0$ ), which is not fulfilled in general. In the case (a) the value of  $\cos(u, v)$  as a value of a free determined limited function can be fixed to zero, i. e.  $\cos(u, v) = 0$ ,  $l_u = 0$ ,  $l_v = 0$ . This value corresponds to the notion orthogonal contravariant non-isotropic (non-null) vector fields. In this way, by definition, the cosine between two isotropic vector fields is equal to zero. On the other side, every isotropic contravariant vector field can be presented as a co-linear to other given isotropic vector field, i.e. if  $u \in T(M)$ ,  $l_u = 0$ ,  $\Rightarrow \exists v \in T(M)$ ,  $l_v = 0$  with  $u = \kappa \cdot v$ ,  $\kappa \in C^r(M)$ . Therefore, under the introduced notion of orthogonality of two isotropic contravariant vector fields the following statement is valid: every two co-linear isotropic contravariant vector fields are orthogonal to each other.*

The cosine change of the angle between two non-isotropic vector fields under the action of the covariant differential operator is determined on the grounds of the definition of the notion cosine and the commutation relation between the

covariant and contraction operator

$$\begin{aligned}
\nabla_\xi \cos(u, v) &= \xi[\cos(u, v)] = \frac{1}{l_u \cdot l_v} [(\nabla_\xi g)(u, v) + g(\nabla_\xi u, v) + g(u, \nabla_\xi v)] - \\
&\quad - [\frac{1}{l_u}(\xi l_u) + \frac{1}{l_v}(\xi l_v)] \cdot \cos(u, v) = \\
&= -[\xi(\log l_u) + \xi(\log l_v)] \cdot \cos(u, v) + \\
&\quad + \frac{1}{l_u \cdot l_v} [(\nabla_\xi g)(u, v) + g(\nabla_\xi u, v) + g(u, \nabla_\xi v)] = \\
&= -[\xi(\log(l_u \cdot l_v))] \cdot \cos(u, v) + \\
&\quad + \frac{1}{l_u \cdot l_v} [(\nabla_\xi g)(u, v) + g(\nabla_\xi u, v) + g(u, \nabla_\xi v)] .
\end{aligned} \tag{13}$$

The last expression can also be written in the form

$$\begin{aligned}
\xi[\cos(u, v)] &= \frac{1}{l_u \cdot l_v} [(\nabla_\xi g)(u, v) + g(\nabla_\xi u, v) + g(u, \nabla_\xi v)] - \\
&\quad - \frac{1}{2} \{ \pm \frac{1}{l_u^2} [(\nabla_\xi g)(u, u) + 2g(\nabla_\xi u, u)] \pm \frac{1}{l_v^2} [(\nabla_\xi g)(v, v) + 2g(\nabla_\xi v, v)] \} \cdot \cos(u, v) .
\end{aligned} \tag{14}$$

The conditions for transports of the covariant metric  $g$  and the conditions for transports of the contravariant vector fields as well determined the change of the cosine of the angle between two contravariant vector fields.

Under the action of the Lie differential operator, the change of the angle between two contravariant vector fields is determined by the action of this operator on a function over manifold and by its commutation relation with the contraction operator

$$\begin{aligned}
\xi[\cos(u, v)] &= \frac{1}{l_u \cdot l_v} [(\mathcal{L}_\xi g)(u, v) + g(\mathcal{L}_\xi u, v) + g(u, \mathcal{L}_\xi v)] - \\
&\quad - [\frac{1}{l_u}(\xi l_u) + \frac{1}{l_v}(\xi l_v)] \cdot \cos(u, v) = \\
&= \frac{1}{l_u \cdot l_v} [(\mathcal{L}_\xi g)(u, v) + g(\mathcal{L}_\xi u, v) + g(u, \mathcal{L}_\xi v)] - \\
&\quad - [\xi(\log l_u) + \xi(\log l_v)] \cdot \cos(u, v) = \\
&= \frac{1}{l_u \cdot l_v} [(\mathcal{L}_\xi g)(u, v) + g(\mathcal{L}_\xi u, v) + g(u, \mathcal{L}_\xi v)] - \\
&\quad - \frac{1}{2} \{ \pm \frac{1}{l_u^2} [(\mathcal{L}_\xi g)(u, u) + 2g(\mathcal{L}_\xi u, u)] \pm \frac{1}{l_v^2} [(\mathcal{L}_\xi g)(v, v) + 2g(\mathcal{L}_\xi v, v)] \} \cdot \cos(u, v) .
\end{aligned} \tag{15}$$

Different draggings-along determine the change of the cosine of the angle between two contravariant non-isotropic vector fields dragged along other contravariant vector field.

The change of the cosine of the angle between two contravariant non-isotropic vector fields under transport or dragging- along is a characteristic of great importance for mathematical models, describing physical systems by means of vector fields over differentiable manifolds with contravariant and covariant affine connections. The geometric characteristics of vector fields (length, angle between two vector fields) have been connected with the characteristics of the physical system and in this way the kinematic structure can be determined for describing physical processes.

## 4 Length of a covariant vector field

The square of the length of a covariant vector field is determined as

$$\bar{g}(p, p) = p^2 = \pm |p|^2 = \pm l_p^2, \quad p \in T^*(M), \quad l_p \geq 0 . \tag{16}$$

By means of this definition the notion length of a covariant vector field is introduced.

**Definition 2** *The length of a covariant vector field  $p$  is the positive square root of the absolute value of the square of the length of this field, i.e.*

$$l_p = |p| = |\bar{g}(p, p)|^{\frac{1}{2}} \quad , \quad l_p(x) = |p_x| = |p(x)| \quad , \quad x \in M \quad , \quad (17)$$

where  $l_p$  is the length of the covariant vector field  $p$  and  $l_p(x)$  is the length of the covariant vector  $p(x)$  in a point  $x \in M$ .

With respect to their lengths the covariant vector fields are also divided in two types: *isotropic (null)*, ( $l_p = 0$ ,  $l_p(x) = 0$ ,  $x \in M$ ), and *non-isotropic (non-null)*, ( $l_p \neq 0$ ,  $l_p(x) \neq 0$ ,  $x \in M$ ), covariant vector fields and vectors. In the cases of definite contravariant metric  $\bar{g}$ , ( $Sgn\bar{g} = n$ ,  $\dim M = n$ ), the isotropic covariant vector field is identically equal to zero ( $p \equiv 0$ ,  $p_\alpha \equiv 0$ ). In the cases of indefinite contravariant metric  $\bar{g}$ , ( $Sgn\bar{g} < n$  or  $Sgn\bar{g} > -n$ ,  $\dim M = n$ ) an isotropic (null) covariant vector field would have different from zero components in an arbitrary basis, i. e. it is not equal to zero in the region where it is defined.

The change of the length of a covariant vector field  $p$  under the action of the covariant differential operator  $\nabla_\xi$  can be found in an analogous way as in the case of contravariant vector field

$$\xi l_p = \pm \frac{1}{2l_p} [(\nabla_\xi \bar{g})(p, p) + 2\bar{g}(\nabla_\xi p, p)] \quad , \quad l_p \neq 0 \quad . \quad (18)$$

The change of the length of a covariant vector field  $p$  under the action of the Lie differential operator  $\mathcal{L}_\xi$  is determined by the expression

$$\xi l_p = \pm \frac{1}{2l_p} [(\mathcal{L}_\xi \bar{g})(p, p) + 2\bar{g}(\nabla_\xi p, p)] \quad . \quad (19)$$

For isotropic (null) covariant vector fields the following relations are fulfilled

$$\begin{aligned} (\nabla_\xi \bar{g})(p, p) &= -2\bar{g}(\nabla_\xi p, p) \quad , \quad l_p = 0 \quad , \\ (\mathcal{L}_\xi \bar{g})(p, p) &= -2\bar{g}(\mathcal{L}_\xi p, p) \quad , \quad l_p = 0 \quad . \end{aligned} \quad (20)$$

Different transports and draggings along of the contravariant metric tensor  $\bar{g}$  induce analogous changes of the covariant vector fields as in the case of contravariant vector fields.

## 5 Cosine of the angle between two covariant vector fields

By means of the scalar product of two covariant vector fields and the length of a covariant vector field, the cosine of the angle between two covariant vector fields is determined by the relation

$$\bar{g}(p, q) = |p| \cdot |q| \cdot \cos(p, q) = l_p \cdot l_q \cdot \cos(p, q) \quad , \quad p, q \in T^*(M) \quad . \quad (21)$$

**Definition 3** *The cosine of the angle between two non-isotropic covariant vector fields  $p$  and  $q$  is by definition*

$$\cos(p, q) = \frac{1}{l_p l_q} \cdot \bar{g}(p, q) , \quad p, q \in T^*(M) , \quad l_p \neq 0 , \quad l_q \neq 0 . \quad (22)$$

**Definition 4** *Two covariant vector fields  $p$  and  $q$  are orthogonal to each other, when*

$$\bar{g}(p, q) = 0 . \quad (23)$$

If the two covariant vector fields  $p$  and  $q$  are null (isotropic) vector fields, then from the last definition (21) it follows that they are orthogonal to each other.

The change of the cosine of the angle between two covariant vector fields under the action of the covariant differential operator  $\nabla_\xi$  can be presented in the form

$$\begin{aligned} \xi[\cos(p, q)] &= \frac{1}{l_p l_q} [(\nabla_\xi \bar{g})(p, q) + \bar{g}(\nabla_\xi p, q) + \bar{g}(p, \nabla_\xi q)] - \\ &\quad - [\xi(\log l_p) + \xi(\log l_q)] \cdot \cos(p, q) = \\ &= \frac{1}{l_p l_q} [(\nabla_\xi \bar{g})(p, q) + \bar{g}(\nabla_\xi p, q) + \bar{g}(p, \nabla_\xi q)] - \\ &\quad - \frac{1}{2} \{ \pm \frac{1}{l_p^2} [(\nabla_\xi \bar{g})(p, p) + 2\bar{g}(\nabla_\xi p, p)] \pm \\ &\quad \pm \frac{1}{l_q^2} [(\nabla_\xi \bar{g})(q, q) + 2\bar{g}(\nabla_\xi q, q)] \} \cdot \cos(p, q) . \end{aligned} \quad (24)$$

The change of the cosine of the angle between two covariant vector fields under the action of the Lie differential operator  $\mathcal{L}_\xi$  can be found in the form

$$\begin{aligned} \xi[\cos(p, q)] &= \frac{1}{l_p l_q} [(\mathcal{L}_\xi \bar{g})(p, q) + \bar{g}(\mathcal{L}_\xi p, q) + \bar{g}(p, \mathcal{L}_\xi q)] - \\ &\quad - [\xi(\log l_p) + \xi(\log l_q)] \cdot \cos(p, q) = \\ &= \frac{1}{l_p l_q} [(\mathcal{L}_\xi \bar{g})(p, q) + \bar{g}(\mathcal{L}_\xi p, q) + \bar{g}(p, \mathcal{L}_\xi q)] - \\ &\quad - \frac{1}{2} \{ \pm \frac{1}{l_p^2} [(\mathcal{L}_\xi \bar{g})(p, p) + 2\bar{g}(\mathcal{L}_\xi p, p)] \pm \\ &\quad \pm \frac{1}{l_q^2} [(\mathcal{L}_\xi \bar{g})(q, q) + 2\bar{g}(\mathcal{L}_\xi q, q)] \} \cdot \cos(p, q) . \end{aligned} \quad (25)$$

The cosine changes of the angle between two non-isotropic covariant vector fields under transports or draggings-along of the contravariant metric can be found in analogous way as in the case of the covariant metric.

## 6 Relative velocity and change of the length of a contravariant vector field

Let us now consider the influence of the kinematic characteristics related to the relative velocity [2] upon the change of the length of a contravariant vector field.

Let  $l_\xi = |g(\xi, \xi)|^{\frac{1}{2}}$  be the length of a contravariant vector field  $\xi$ . The rate of change  $ul_\xi$  of  $l_\xi$  along a contravariant vector field  $u$  can be expressed in the form  $\pm 2.l_\xi \cdot (ul_\xi) = (\nabla_u g)(\xi, \xi) + 2g(\nabla_u \xi, \xi)$ . By the use of the projections of  $\xi$  and

$\nabla_u \xi$  along and orthogonal to  $u$  (see the chapter about kinematic characteristics and relative velocity) we can find the relations

$$\begin{aligned} 2g(\nabla_u \xi, \xi) &= 2 \cdot \frac{l}{e} \cdot g(\nabla_u \xi, u) + 2g(\text{rel}v, \xi_\perp) , \\ (\nabla_u g)(\xi, \xi) &= (\nabla_u g)(\xi_\perp, \xi_\perp) + 2 \cdot \frac{l}{e} \cdot (\nabla_u g)(\xi_\perp, u) + \frac{l^2}{e^2} \cdot (\nabla_u g)(u, u) . \end{aligned}$$

Then, it follows for  $\pm 2.l_\xi.(ul_\xi)$  the expression

$$\begin{aligned} \pm 2.l_\xi.(ul_\xi) &= (\nabla_u g)(\xi_\perp, \xi_\perp) + 2 \cdot \frac{l}{e} \cdot [(\nabla_u g)(\xi_\perp, u) + g(\nabla_u \xi, u)] + \\ &\quad + \frac{l^2}{e^2} \cdot (\nabla_u g)(u, u) + 2g(\text{rel}v, \xi_\perp) , \end{aligned} \quad (26)$$

where

$$g(\text{rel}v, \xi_\perp) = \frac{l}{e} \cdot h_u(a, \xi_\perp) + h_u(\mathcal{L}_u \xi, \xi_\perp) + d(\xi_\perp, \xi_\perp) , \quad (27)$$

$$d(\xi_\perp, \xi_\perp) = \sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot l_{\xi_\perp}^2 . \quad (28)$$

For finding out the last two expressions the following relations have been used:

$$g(\overline{g}(h_u)a, \xi_\perp) = h_u(a, \xi_\perp) , \quad g(\overline{g}(h_u)(\mathcal{L}_u \xi), \xi_\perp) = h_u(\mathcal{L}_u \xi, \xi_\perp) , \quad (29)$$

$$g(\overline{g}[d(\xi)], \xi_\perp) = d(\xi_\perp, \xi_\perp) , \quad d(\xi) = d(\xi_\perp) . \quad (30)$$

*Special case:*  $g(u, \xi) = l := 0 : \xi = \xi_\perp$ .

$$\pm 2.l_{\xi_\perp} \cdot (ul_{\xi_\perp}) = (\nabla_u g)(\xi_\perp, \xi_\perp) + 2g(\text{rel}v, \xi_\perp) . \quad (31)$$

*Special case:*  $V_n$ -spaces:  $\nabla_\eta g = 0$  for  $\forall \eta \in T(M)$  ( $g_{ij;k} = 0$ ),  $g(u, \xi) = l := 0 : \xi = \xi_\perp$ .

$$\pm l_{\xi_\perp} \cdot (ul_{\xi_\perp}) = g(\text{rel}v, \xi_\perp) . \quad (32)$$

In  $(\overline{L}_n, g)$ -spaces as well as in  $(L_n, g)$ -spaces the covariant derivative  $\nabla_u g$  of the metric tensor field  $g$  along  $u$  can be decomposed in its trace free part  ${}^s\nabla_u g$  and its trace part  $\frac{1}{n} \cdot Q_u \cdot g$  as

$$\nabla_u g = {}^s\nabla_u g + \frac{1}{n} \cdot Q_u \cdot g , \quad \dim M = n ,$$

where

$$\overline{g}[{}^s\nabla_u g] = 0 , \quad Q_u = \overline{g}[\nabla_u g] = g^{\overline{k}l} \cdot g_{kl;j} \cdot u^j = Q_j \cdot u^j , \quad Q_j = g^{\overline{k}l} \cdot g_{kl;j} .$$

The covariant vector  $\overline{Q} = \frac{1}{n} \cdot Q = \frac{1}{n} \cdot Q_j \cdot dx^j = \frac{1}{n} \cdot Q_\alpha \cdot e^\alpha$  is called *Weyl's covector field*. The operator  $\nabla_u = {}^s\nabla_u + \frac{1}{n} \cdot Q_u$  is called *trace free covariant operator*.

If we use now the decomposition of  $\nabla_u g$  in the expression for  $\pm 2.l_\xi.(ul_\xi)$  we find the relation

$$\begin{aligned} \pm 2.l_\xi.(ul_\xi) &= ({}^s\nabla_u g)(\xi, \xi) + \frac{1}{n} \cdot Q_u \cdot l_\xi^2 + 2g(\nabla_u \xi, \xi) = \\ &= ({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \\ &+ \frac{l}{e} \cdot [2 \cdot ({}^s\nabla_u g)(\xi_\perp, u) + 2 \cdot g(\nabla_u \xi, u) + \frac{l}{e} \cdot ({}^s\nabla_u g)(u, u)] + \\ &\quad + \frac{1}{n} \cdot Q_u \cdot (l_{\xi_\perp}^2 + \frac{l^2}{e}) + 2 \cdot g(\text{rel}v, \xi_\perp) , \end{aligned} \quad (33)$$

where  $l_{\xi_\perp}^2 = g(\xi_\perp, \xi_\perp)$ ,  $l = g(\xi, u)$ .  
For  $l_\xi \neq 0$  :

$$ul_\xi = \pm \frac{1}{2.l_\xi} ({}^s\nabla_u g)(\xi, \xi) \pm \frac{1}{2.n} . Q_u . l_\xi \pm \frac{1}{l_\xi} . g(\nabla_u \xi, \xi) . \quad (34)$$

In the case of a parallel transport ( $\nabla_u \xi = 0$ ) of  $\xi$  along  $u$  the change  $ul_\xi$  of the length  $l_\xi$  is

$$ul_\xi = \pm \frac{1}{2.l_\xi} ({}^s\nabla_u g)(\xi, \xi) \pm \frac{1}{2.n} . Q_u . l_\xi . \quad (35)$$

*Special case:*  $\nabla_u \xi = 0$  and  ${}^s\nabla_u g = 0$ .

$$ul_\xi = \pm \frac{1}{2.n} . Q_u . l_\xi . \quad (36)$$

If  $u = \frac{d}{ds} = u^i . \partial_i = (dx^i/ds) . \partial_i$ , then

$$\begin{aligned} l_\xi(s+ds) &\approx l_\xi(s) + \frac{dl_\xi}{ds} . ds = l_\xi(s) \pm \frac{1}{2.n} . Q_u(s) . l_\xi(s) . ds = \\ &= (1 \pm \frac{1}{2.n} . Q_u(s) . ds) . l_\xi(s) = \triangle_u(s) . l_\xi(s) , \\ \triangle_u(s) &= 1 \pm \frac{1}{2.n} . Q_u(s) . ds . \end{aligned} \quad (37)$$

Therefore, the rate of change of  $l_\xi$  along  $u$  is linear to  $l_\xi$ .

*Special case:*  $g(u, \xi) = l := 0 : \xi = \xi_\perp$ .

$$\begin{aligned} \pm 2.l_{\xi_\perp} . (ul_{\xi_\perp}) &= ({}^s\nabla_u g)(\xi_\perp, \xi_\perp) + \frac{1}{n} . Q_u . l_{\xi_\perp}^2 + 2.g(\text{rel}v, \xi_\perp) . \\ ul_{\xi_\perp} &= \pm \frac{1}{2.l_{\xi_\perp}} . ({}^s\nabla_u g)(\xi_\perp, \xi_\perp) \pm \frac{1}{2n} . Q_u . l_{\xi_\perp} \pm \frac{1}{l_{\xi_\perp}} . g(\text{rel}v, \xi_\perp) , \quad l_{\xi_\perp} \neq 0 . \end{aligned} \quad (38)$$

*Special case:* Quasi-metric transports:  $\nabla_u g := 2.g(u, \eta) . g$ ,  $u, \eta \in T(M)$ .

$$\pm 2.l_\xi . (ul_\xi) = 2.g(u, \eta) . (l_{\xi_\perp}^2 + \frac{l^2}{e}) + 2.[\frac{l}{e} . g(\nabla_u \xi, u) + g(\text{rel}v, \xi_\perp)] . \quad (39)$$

## 7 Change of the cosine between two contravariant vector fields and the relative velocity

The cosine between two contravariant vector fields  $\xi$  and  $\eta$  has been defined as  $g(\xi, \eta) = l_\xi . l_\eta . \cos(\xi, \eta)$ . The rate of change of the cosine along a contravariant vector field  $u$  can be found in the form

$$\begin{aligned} l_\xi . l_\eta . \{u[\cos(\xi, \eta)]\} &= (\nabla_u g)(\xi, \eta) + g(\nabla_u \xi, \eta) + g(\xi, \nabla_u \eta) - \\ &\quad - [l_\eta . (ul_\xi) + l_\xi . (ul_\eta)] . \cos(\xi, \eta) . \end{aligned}$$

*Special case:*  $\nabla_u \xi = 0, \nabla_u \eta = 0, {}^s\nabla_u g = 0$ .

$$l_\xi \cdot l_\eta \cdot \{u[\cos(\xi, \eta)]\} = \frac{1}{n} \cdot Q_u \cdot g(\xi, \eta) - [l_\eta \cdot (ul_\xi) + l_\xi \cdot (ul_\eta)] \cdot \cos(\xi, \eta) .$$

Since  $g(\xi, \eta) = l_\xi \cdot l_\eta \cdot \cos(\xi, \eta)$ , it follows from the last relation

$$l_\xi \cdot l_\eta \cdot \{u[\cos(\xi, \eta)]\} = \left\{ \frac{1}{n} \cdot Q_u \cdot l_\xi \cdot l_\eta - [l_\eta \cdot (ul_\xi) + l_\xi \cdot (ul_\eta)] \right\} \cdot \cos(\xi, \eta) .$$

Therefore, if  $\cos(\xi, \eta) = 0$  between two parallel transported along  $u$  vector fields  $\xi$  and  $\eta$ , then the right angle between them [determined by the condition  $\cos(\xi, \eta) = 0$ ] does not change along the contravariant vector field  $u$ . In the cases, when  $\cos(\xi, \eta) \neq 0$ , the rate of change of the cosine of the angle between two vector fields  $\xi$  and  $\eta$  is linear to  $\cos(\xi, \eta)$ .

By the use of the definitions and the relations:

$${}_{rel}v_\xi := \overline{g}[h_u(\nabla_u \xi)] = {}_{rel}v , \quad {}_{rel}v_\eta := \overline{g}[h_u(\nabla_u \eta)] , \quad (40)$$

$$\begin{aligned} g(\nabla_u \xi, \eta) &= \frac{1}{e} \cdot g(u, \eta) \cdot g(\nabla_u \xi, u) + g({}_{rel}v_\xi, \eta) , \\ g(\nabla_u \eta, \xi) &= \frac{1}{e} \cdot g(u, \xi) \cdot g(\nabla_u \eta, u) + g({}_{rel}v_\eta, \xi) , \end{aligned} \quad (41)$$

$$(\nabla_u g)(\xi, \eta) = ({}^s\nabla_u g)(\xi, \eta) + \frac{1}{n} \cdot Q_u \cdot g(\xi, \eta) , \quad (42)$$

$$\begin{aligned} ({}^s\nabla_u g)(\xi, \eta) &= ({}^s\nabla_u g)(\xi_\perp, \eta_\perp) + \frac{l}{e} \cdot ({}^s\nabla_u g)(u, \eta_\perp) + \frac{\bar{l}}{e} \cdot ({}^s\nabla_u g)(\xi_\perp, u) + \\ &+ \frac{l}{e} \cdot \frac{\bar{l}}{e} \cdot ({}^s\nabla_u g)(u, u) , \quad \bar{l} = g(u, \eta) , \quad \eta_\perp = \overline{g}[h_u(\eta)] , \quad l = g(u, \xi) , \end{aligned} \quad (43)$$

$$\begin{aligned} (\nabla_u g)(\xi, \eta) &= ({}^s\nabla_u g)(\xi, \eta) + \frac{1}{n} \cdot Q_u \cdot g(\xi, \eta) = \\ &= ({}^s\nabla_u g)(\xi_\perp, \eta_\perp) + \frac{l}{e} \cdot ({}^s\nabla_u g)(u, \eta_\perp) + \frac{\bar{l}}{e} \cdot ({}^s\nabla_u g)(\xi_\perp, u) + \\ &+ \frac{l}{e} \cdot \frac{\bar{l}}{e} \cdot ({}^s\nabla_u g)(u, u) + \frac{1}{n} \cdot Q_u \cdot \left[ \frac{l\bar{l}}{e} + g(\xi_\perp, \eta_\perp) \right] , \end{aligned} \quad (44)$$

the expression of  $l_\xi \cdot l_\eta \cdot \{u[\cos(\xi, \eta)]\}$  follows in the form

$$\begin{aligned} l_\xi \cdot l_\eta \cdot \{u[\cos(\xi, \eta)]\} &= ({}^s\nabla_u g)(\xi_\perp, \eta_\perp) + \frac{l}{e} \cdot [({}^s\nabla_u g)(u, \eta_\perp) + g(\nabla_u \eta, u)] + \\ &+ \frac{\bar{l}}{e} \cdot [({}^s\nabla_u g)(\xi_\perp, u) + g(\nabla_u \xi, u)] + \frac{l\bar{l}}{e^2} \cdot ({}^s\nabla_u g)(u, u) + \\ &+ \frac{1}{n} \cdot Q_u \cdot \left[ \frac{l\bar{l}}{e} + g(\xi_\perp, \eta_\perp) \right] + g({}_{rel}v_\xi, \eta) + g({}_{rel}v_\eta, \xi) - \\ &- [l_\eta \cdot (ul_\xi) + l_\xi \cdot (ul_\eta)] \cdot \cos(\xi, \eta) . \end{aligned} \quad (45)$$

*Special case:*  $g(u, \xi) = l := 0, g(u, \eta) = \bar{l} := 0 : \xi = \xi_\perp, \eta = \eta_\perp$ .

$$\begin{aligned} l_{\xi_\perp} \cdot l_{\eta_\perp} \cdot \{u[\cos(\xi_\perp, \eta_\perp)]\} &= ({}^s\nabla_u g)(\xi_\perp, \eta_\perp) + \frac{1}{n} \cdot Q_u \cdot l_{\xi_\perp} \cdot l_{\eta_\perp} \cdot \cos(\xi_\perp, \eta_\perp) + \\ &+ g({}_{rel}v_{\xi_\perp}, \eta_\perp) + g({}_{rel}v_{\eta_\perp}, \xi_\perp) - [l_{\eta_\perp} \cdot (ul_{\xi_\perp}) + l_{\xi_\perp} \cdot (ul_{\eta_\perp})] \cdot \cos(\xi_\perp, \eta_\perp) , \end{aligned} \quad (46)$$

where  $g(\xi_\perp, \eta_\perp) = l_{\xi_\perp} \cdot l_{\eta_\perp} \cdot \cos(\xi_\perp, \eta_\perp)$ .

The kinematic characteristics related to the relative velocity and used in considerations of the rate of change of the length of contravariant vector fields as well as the change of the angle between two contravariant vector fields could also be useful for description of the motion of physical systems in  $(\overline{L}_n, g)$ -spaces.

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